A Comparative Analysis of Hyperbolic Copulas Induced by a One Factor Lévy Model

August 20, 2006

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Abstract

In the credit derivatives market, the observed default correlation smile, implied by the Gaussian copula, constitutes a major problem when we want to price bespoke CDO tranches. The industry standard approach for countering this dilemma is to use the concept of base correlation to try to estimate the ingoing default correlation parameters for non-standard tranche intervals. However, this approach is far from mathematically consistent and the method of interpolating between different points on the base correlation curve is all but accurate. In this paper, we generalize the Gaussian one factor model proposed by Vasicek (1987) to work with any underlying Lévy process. Further, we show that our one Factor Lévy model induces an infinite number of different copulas. Given this mathematical framework, we propose three different hyperbolic copulas, namely a normal inverse Gaussian copula, a variance gamma copula and a skewed Student t copula. In addition to calculate the loss distributions by $\mathcal{FFT}$ methods, we generalize the LHP approximation given by Vasicek (1991) to work with our Lévy model. Finally, we show that the proposed alternative copulas give superior results and we give a comparative view on the different methods for calculating the loss distributions.

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1 Introduction

The first collateralized debt obligations (CDOs) were structured out of pools of high yield bonds and mortgages in the late 1980s. Later, as the credit default swap (CDS) market became more liquid and standardized in the 1990s, the first synthetic CDOs were introduced by JPMorgan and the Swiss Bank Corporation in 1997. Today, this type of CDO structure makes up the majority of the market size and the total trading volume has surpassed that of corporate bonds.

The de facto industry standard approach for pricing CDOs is the Gaussian copula model, where the correlation structure is set up by the one factor model, first proposed by Vasicek (1987). The great advantage of this latent factor approach is that the dimensionality problem of calculating the aggregate loss distribution is drastically reduced. However, as has been shown in the literature, the Gaussian copula does a poor job at explaining the market prices. In particular, if the Gaussian copula is fitted to the traded tranches of a CDO, it produces a distinct correlation smile. This smile forms a major problem if we want to price non-standard CDO tranches. The industry has mainly used the concept of base correlation for bespoke pricing but that approach is inconsistent and has shown to yield unsatisfactory results. A more adequate methodology would be to choose another copula that describes the default dependence in a more realistic manner.

Burtschell et al. (2005) present a comparative analysis of Student t, Clayton and Marshall-Olkin copulas, where the double t copula generates the best market fit. However, more recent proposals using hyperbolic distributions have shown to give even better results. Kalemanova et al. (2005) and Guegan and Houdain (2005) propose two different types of normal inverse Gaussian ($\mathcal{NIG}$) copulas and Moosbrucker (2005) propose a variance gamma ($\mathcal{VG}$) copula. These papers all show that skewness is of great importance, indicating that the dependence of negative default events are higher than the positive ones.

In this paper we generalize the structural approach made by Vasicek (1987) to work with any kind of underlying Lévy process. We investigate on the set of possible copulas induced by our one factor Lévy model and propose a skewed Student t ($\mathcal{S}t$) copula in addition to a $\mathcal{NIG}$ and $\mathcal{VG}$ copula. Further, we deduce the LHP approximation for our Lévy setup, generalizing the result in Vasicek (1991). Loss distributions are calculated using both LHP approximations and Fourier methods, investigating whether the Fourier based approach, which in theory should be more exact, yields better results.

The rest of this paper will be organized as follows. In section 2, we present the one factor Lévy model. In section 3, we derive the possible set of copulas given our model and set up the battery of hyperbolic copulas that will be used for market calibration. In section 4, exact and approximative methods for loss distribution calculation will be presented. In section 5, we give the arbitrage relations for CDO tranche pricing. In section 6, the proposed models will be calibrated to current market data. Concluding remarks close the paper.

Preliminaries

We shall assume a complete and filtered probability space of the form $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$, which satisfies the usual conditions and where $\mathbb{F} = \{\mathcal{F}_t, t \geqslant 0\}$. Throughout the paper, we will fix an equivalent martingale measure $\mathbb{Q}$ directly on our probability space and we shall assume that such a measure always exists. Further, we will not worry about market incompleteness and simply focus on pricing issues. When modelling credit portfolio risk, we shall assume that all stopping times are totally inaccessible, i.e. defaults cannot occur simultaneously.

2 A One Factor Lévy Model

Vasicek (1987) introduced the one factor model as a direct extension to the structural firm value model presented by Merton (1974). The firm value process in Merton's model follows a geometric brownian motion and the value of an entity i at time t is thus given by

$$A_{t,i} = A_{0,i} \exp\left\{ \left(r - \frac{\sigma^2}{2} \right) t + \sigma W_t \right\}, \tag{1}$$

where r and σ are known constants and W_t follows a Wiener process. To make a more general approach, we shall instead assume that $A_{t,i}$ is given by

$$A_{t,i} = A_{0,i} \exp\left\{ X_t \right\}, \tag{2}$$

where $(X_t)_{t \in \mathbb{R}^+}$ is a Lévy process with finite variance. Accordingly, given some default barrier D_i, the probability of default for entity i is given by

$$p^d_{t,i} = \mathbb{Q}\left[X_{t,i} < \ln\left(\frac{D_i}{A_{0,i}} \right) \right] = F_{X_i}\left(\frac{\ln\left(\frac{D_i}{A_{0,i}} \right)}{\sqrt{t}} \right), \tag{3}$$

where F_{X_i} denotes the cumulative distribution function of a normalized stationary increment, i.e.

$$X_i = \frac{X_{t+s} - X_t}{\sqrt{s}}.[1] \tag{4}$$

Now, to introduce a correlation structure into the model, we choose to represent the Lévy process by a factor model given by

$$X_{t,i} = \alpha M_t + \beta \epsilon_{t,i}, \tag{5}$$

where M_t represents the systematic market risk and $\epsilon_{t,i}$ represents the idiosyncratic risk for each entity. For simplicity we assume that the correlation factors are constant

[1] Hereafter, whenever the subindex t is excluded from a stochastic process, we will refer to a normalized stationary increment.

over time and that the correlation is the same for any pair of credits, e.i. $\rho_{t,i,j} = \rho$ for all i, j. Accordingly, we must have that $\alpha^2 + \beta^2 = 1$ and by Itô calculus we obtain

$$\mathrm{E}[dX_{t,i}dX_{t,j}] = \alpha^2\sigma^2 dt = \rho\sigma^2 dt, \quad i \neq j \qquad \Rightarrow \qquad \alpha = \sqrt{\rho} \tag{6a}$$
$$\mathrm{E}[(dX_{t,i})^2] = (\alpha^2 + \beta^2)\sigma^2 dt = \sigma^2 dt \qquad \Rightarrow \qquad \beta = \sqrt{1-\rho}, \tag{6b}$$

which finally gives us that equation (5) can be written as

$$X_{t,i} = \sqrt{\rho} M_t + \sqrt{1-\rho}\,\epsilon_{t,i}. \tag{7}$$

□

Given the above equation, we can now express the probability of default conditioned on the common factor by

$$p_{t,i}^{d|M} = F_{\epsilon_i}\left(\frac{F_{X_i}^{-1}\left(p_{t,i}^d\right) - \sqrt{\rho}M}{\sqrt{1-\rho}}\right). \tag{8}$$

This setup greatly reduces the dimensionality problem of calculating joint distributions of a credit portfolio, since the default probabilities conditioned on the common factor are mutually independent. In particular, it provides us with a method to get univariate marginal distribution functions that can be used in a copula.

3 Copulas Induced by the Lévy One Factor Model

As Vasicek's one factor model results in a Gaussian copula, the more general approach of a Lévy factor model provides us with an endless variety of different copulas. To understand the extent of the class of copulas that we can choose from, we start by presenting some characteristics of the Lévy process.

For every infinitely divisible distribution X, we can define a Lévy process. The distribution of X is said to be infinitely divisible if we can find a sum of i.i.d. variables such that $X = \sum_{i=1}^{n} Z_i$, $n \in \mathbb{N}$. Further, the sum of two infinitely divisible distributions is also infinitely divisible, see Sato (1999). Alas, the only restrictions to our Lévy factor copula are that M and ϵ_i need to be infinitely divisible distributions with zero mean and equal finite variance.

3.1 Generalized Hyperbolic Copulas

The generalized hyperbolic ($\mathcal{GH}$) distribution was introduced by Barndorff-Nielsen (1977b) for describing dune movements and was first applied to financial time series by Eberlein and Keller (1995). Today, the $\mathcal{GH}$ distribution and its subclasses are very popular within the finance field since they have proven to be able to give almost exact fits to different log returns, see e.g. Prause (1999). Barndorff-Nielsen (1977a) proves that the $\mathcal{GH}$ distribution is infinitely divisible and thus induce a Lévy process. These facts motivate using hyperbolic distributions for correlated default modelling.

All hyperbolic distributions can be deduced as subclasses from the $\mathcal{GH}$ distribution, which density function is given by

$$\begin{aligned} f(x;\lambda,\alpha,\beta,\delta,\mu)_{\mathcal{GH}} =& a(\lambda,\alpha,\beta,\delta,\mu)\left(\delta^2+(x-\mu)^2\right)^{\left(\lambda-\frac{1}{2}\right)/2} \\ &\times K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)\exp\left\{\beta(x-\mu)\right\}, \end{aligned} \tag{9a}$$

where

$$a(\lambda,\alpha,\beta,\delta,\mu)=\frac{\left(\alpha^2-\beta^2\right)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}K_{\lambda}\left(\delta\sqrt{\alpha^2-\beta^2}\right)} \tag{9b}$$

and $K_\lambda(\cdot)$ denotes the modified Bessel function of the third kind, order λ, given by

$$K_\lambda(x)=\frac{1}{2}\int_0^\infty y^{\lambda-1}\exp\left\{-\frac{x}{2}\left(y+y^{-1}\right)\right\}dy, \quad x>0. \tag{9c}$$

The parameter restrictions are

$$\begin{aligned} &\mu\in\mathbb{R} \\ &\delta\geqslant 0, |\beta|<\alpha \quad if \quad \delta>0 \\ &\delta>0, |\beta|<\alpha \quad if \quad \delta=0 \\ &\delta>0, |\beta|\leqslant\alpha \quad if \quad \delta<0, \end{aligned} \tag{10}$$

where δ and μ describes the scale and location respectively, β describes the skewness and a decrease in $\delta\sqrt{\alpha^2-\beta^2}$ reflects an increase in kurtosis. Further, the moment generating function and characteristic function are shown to be

$$\varphi(u)_{\mathcal{GH}}=e^{u\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\lambda/2}\frac{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+u)^2}\right)}{K_\lambda\left(\delta\sqrt{\alpha^2-\beta^2}\right)}, |\beta+u|<\alpha \tag{11}$$

$$\phi(u)_{\mathcal{GH}}=\varphi(iu)_{\mathcal{GH}} \tag{12}$$

respectively. We note that $\phi(u)^t_{\mathcal{GH}}$ in general have the form of equation (12) only if $t=1$. Thus, in general, the $\mathcal{GH}$ distribution is not stable under convolution. The only known exceptions are the normal inverse Gaussian ($\mathcal{NIG}$) distribution and the variance gamma ($\mathcal{VG}$) distribution. Another important characteristic of the $\mathcal{GH}$ distribution is that is has semi-heavy tails, see Barndorff-Nielsen and Blæsild (1981), which behaves as

$$f(x;\lambda,\alpha,\beta,\delta,\mu)_{\mathcal{GH}}\sim|x|^{\lambda-1}\exp\left\{(\mp\alpha+\beta)x\right\} \quad as \quad x\to\pm\infty \tag{13}$$

up to a multiplicative constant. This allows for greater tail-dependence in our copula setup, and as such, enables us to model the dependence of extreme default events. Before we take a closer look at the subclasses that will we used in this paper, we'd like to mention that the standard Gaussian copula can be obtained as a limiting subclass of the $\mathcal{GH}$ copulas since

$$\lim_{\delta\to\infty,\delta/\alpha\to\sigma^2}\mathcal{GH}(\lambda,\alpha,\beta,\delta,\mu)\sim\mathcal{N}(\mu+\beta\sigma^2,\sigma^2). \tag{14}$$

For a thorough presentation on limiting behavior of the $\mathcal{GH}$ distribution, we refer to Eberlein and Hammerstein (2002).

3.1.1 A Normal Inverse Gaussian Copula

The $\mathcal{NIG}$ distribution was introduced into finance by Barndorff-Nielsen (1997) and its subclass is given by

$$\mathcal{NIG}\,(\alpha, \beta, \delta, \mu) \sim \mathcal{GH}(-\frac{1}{2}, \alpha, \beta, \delta, \mu). \tag{15}$$

Taking moments yields that

$$\begin{aligned}
\mathrm{E}[X] &= \mu + \frac{\beta\delta}{\sqrt{\alpha^2 - \beta^2}} \\
\mathrm{V}[X] &= \frac{\delta\alpha^2}{(\alpha^2 - \beta^2)^{3/2}} \\
\mathrm{S}[X] &= \frac{3\beta}{\alpha\sqrt{\delta}\,(\alpha^2 - \beta^2)^{1/4}} \\
\mathrm{K}[X] &= \frac{3\left(1 + 4\left(\frac{\beta}{\alpha}\right)^2\right)}{\delta\sqrt{\alpha^2 - \beta^2}}.
\end{aligned}$$

A strong argument for using the $\mathcal{NIG}$ distribution with the one factor Lévy copula is its closeness under convolution property, given by

$$\begin{aligned}
&\begin{cases} X_1 \sim \mathcal{NIG}(\alpha_1, \beta_1, \mu_1, \delta_1) \\ X_2 \sim \mathcal{NIG}(\alpha_2, \beta_2, \mu_2, \delta_2) \\ \alpha_1 = \alpha_2 = \alpha, \quad \beta_1 = \beta_2 = \beta \end{cases} \\
&\Rightarrow X_1 + X_2 \sim \mathcal{NIG}(\alpha, \beta, \mu_1 + \mu_2, \delta_1 + \delta_2),
\end{aligned} \tag{17}$$

Further, its scaling property is given by

$$\begin{aligned}
X &\sim \mathcal{NIG}(\alpha, \beta, \mu, \delta) \\
\Rightarrow cX &\sim \mathcal{NIG}\left(\frac{\alpha}{c}, \frac{\beta}{c}, c\mu, c\delta\right).
\end{aligned} \tag{18}$$

Accordingly, given the right choice of parameterization, we can reduce the number of free variables and get an analytically tractable copula setup.

We start by defining the systematic risk factor as

$$M \sim \mathcal{NIG}\left(\alpha, \beta, \frac{\left(\alpha^2 - \beta^2\right)^{3/2}}{\alpha^2}, -\frac{\beta\left(\alpha^2 - \beta^2\right)}{\alpha^2}\right), \tag{19}$$

where δ_M and μ_M is chosen so that M gets zero mean and unit variance. Keeping X_i $\mathcal{NIG}$ distributed, we then get that

$$\begin{aligned}
\epsilon_i \sim \mathcal{NIG}\Bigg(&\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\beta \\
&, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\left(\alpha^2 - \beta^2\right)^{3/2}}{\alpha^2}, -\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\frac{\beta\left(\alpha^2 - \beta^2\right)}{\alpha^2}\Bigg)
\end{aligned} \tag{20a}$$

$$X_i \sim \mathcal{NIG}\left(\frac{1}{\sqrt{\rho}}\alpha, \frac{1}{\sqrt{\rho}}\beta, \frac{1}{\sqrt{\rho}}\frac{\left(\alpha^2-\beta^2\right)^{3/2}}{\alpha^2}, -\frac{1}{\sqrt{\rho}}\frac{\beta\left(\alpha^2-\beta^2\right)}{\alpha^2}\right), \tag{20b}$$

3.1.2 A Variance Gamma Copula

The $\mathcal{VG}$ distribution was introduced by Madan and Seneta (1990) and its subclass is given by

$$\mathcal{VG}\left(\lambda,\alpha,\beta,\mu\right) \sim \mathcal{GH}\left(\lambda,\alpha,\beta,0,\mu\right)^{2} \tag{21}$$

Taking moments yields that

$$\begin{aligned}
\mathrm{E}[X] &= \mu + \frac{2\beta\lambda}{\alpha^2-\beta^2}\\
\mathrm{V}[X] &= \frac{2\lambda\left(\alpha^2+\beta^2\right)}{\left(\alpha^2-\beta^2\right)^2}\\
\mathrm{S}[X] &= \frac{\sqrt{2}\beta\left(3\alpha^2+\beta^2\right)}{\sqrt{\frac{\lambda(\alpha^2+\beta^2)}{(\alpha^2-\beta^2)^2}}\left(\alpha^4-\beta^4\right)}\\
\mathrm{K}[X] &= \frac{3\left(\lambda\alpha^4+2\lambda\alpha^2\beta^2+\lambda\beta^4+\alpha^4+6\alpha^2\beta^2+\beta^4\right)}{\lambda\left(\alpha^2+\beta^2\right)^2}.
\end{aligned}$$

Its convolution property is given by

$$\begin{aligned}
&\begin{cases} X_1 \sim \mathcal{VG}\left(\lambda_1,\alpha_1,\beta_1,\mu_1\right)\\ X_2 \sim \mathcal{VG}\left(\lambda_2,\alpha_2,\beta_2,\mu_2\right)\\ \alpha_1=\alpha_2=\alpha, \quad \beta_1=\beta_2=\beta\end{cases}\\
&\Rightarrow X_1+X_2 \sim \mathcal{VG}\left(\lambda_1+\lambda_2,\alpha,\beta,\mu_1+\mu_2\right)
\end{aligned} \tag{22}$$

and its scaling property by

$$\begin{aligned}
X &\sim \mathcal{VG}\left(\lambda,\alpha,\beta,\mu\right)\\
\Rightarrow cX &\sim \mathcal{VG}\left(\lambda,\frac{\alpha}{c},\frac{\beta}{c},c\mu\right).
\end{aligned} \tag{23}$$

Being the second subclass that is closed under convolution, the $\mathcal{VG}$ distribution is a good candidate for our Lévy one factor copula.

We start by defining the systematic risk factor as

$$M \sim \mathcal{VG}\left(\frac{\left(\alpha^2-\beta^2\right)^2}{2\left(\alpha^2+\beta^2\right)},\alpha,\beta,-\beta\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\right), \tag{24}$$

[2] The alternative notation $\mathcal{VG}\left(\theta,\nu,\sigma,\bar{\mu}\right)$ is also common in the literature. This parameterization is obtained by doing the following substitutions; $\sigma^2=\frac{2\lambda}{\alpha^2-\beta^2}$, $\quad \nu=\frac{1}{\lambda}$, $\quad \theta=\beta\sigma^2$, $\quad \bar{\mu}=\mu$.

where δ_M and μ_M is set so to get zero mean and unit variance. Thus, keeping X_i $\mathcal{VG}$ distributed, we get that

$$\epsilon_i \sim \mathcal{VG}\left(\frac{1-\rho}{\rho}\frac{\left(\alpha^2-\beta^2\right)^2}{2\left(\alpha^2+\beta^2\right)}, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\alpha, \frac{\sqrt{1-\rho}}{\sqrt{\rho}}\beta, -\frac{\sqrt{1-\rho}}{\sqrt{\rho}}\beta\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\right) \tag{25a}$$

$$X_i \sim \mathcal{VG}\left(\frac{1}{\rho}\frac{\left(\alpha^2-\beta^2\right)^2}{2\left(\alpha^2+\beta^2\right)}, \frac{1}{\sqrt{\rho}}\alpha, \frac{1}{\sqrt{\rho}}\beta, -\frac{1}{\sqrt{\rho}}\beta\frac{\alpha^2-\beta^2}{\alpha^2+\beta^2}\right). \tag{25b}$$

3.1.3 A Skewed Student t Copula

Both the $\mathcal{NIG}$ distribution and the $\mathcal{VG}$ distribution have semi-heavy tails and can be modified to give skewness. However, if the financial data has heavy tails, it might be desirable to use a distribution with polynomial tails. The Student t distribution has polynomial tails but it can't be modified to give skewness. Furthermore, empirical studies indicate that it is often preferable to have only one tail polynomial and keep the second exponential. The $\mathcal{S}t$ distribution, introduced by Aas and Haff (2006), can model both skewness as well as having one polynomial tail and one exponential tail. It is the only subclass that possess these properties and it is given by

$$\mathcal{S}t\left(\nu,\beta,\delta,\mu\right) \sim \mathcal{GH}\left(-\frac{\nu}{2}, |\beta|, \beta, \delta, \mu\right). \tag{26}$$

Taking moments yields that

$$\begin{aligned}
\mathrm{E}[X] &= \mu + \frac{\beta\delta^2}{\nu-2} \\
\mathrm{V}[X] &= \frac{2\beta^2\delta^4}{(\nu-2)^2(\nu-4)} + \frac{\delta^2}{\nu-2}, \quad \nu > 4 \\
\mathrm{S}[X] &= \frac{2(\nu-4)^{1/2}\beta\delta}{\left(2\beta^2\delta^2+(\nu-2)(\nu-4)\right)^{3/2}}\left(3(\nu-2)+\frac{8\beta^2\delta^2}{\nu-6}\right), \quad \nu > 7 \\
\mathrm{K}[X] &= \frac{6}{\left(2\beta^2\delta^2+(\nu-2)(\nu-4)\right)^2}\Bigg((\nu-2)^2(\nu-4) \\
&\quad + \frac{16\beta^2\delta^2(\nu-2)(\nu-4)}{\nu-6} + \frac{8\beta^4\delta^4(5\nu-22)}{(\nu-6)(\nu-8)}\Bigg), \quad \nu > 9.
\end{aligned}$$

The disadvantage of the $\mathcal{S}t$ distribution is that it is not stable under convolution. Thus, any convolution must be calculated numerically. Secondly, letting both the systematic and idiosyncratic risk factors be $\mathcal{S}t$ distributed would yield many free variables, making the copula impractical due to optimization issues. Therefore, we propose a copula where the systematic risk factor follows a $\mathcal{S}t$ distribution and the idiosyncratic risk factor follows a Gaussian distribution. We let

$$M \sim \mathcal{S}t\left(\nu,\beta,\delta_M,\mu_M\right) \tag{28a}$$

$$\epsilon_i \sim \mathcal{N}(0,1), \tag{28b}$$

where δ_M and μ_M are set so that our systematic risk factor has zero mean and unit variance, e.i.

$$\delta_M = \frac{1}{2\beta}\sqrt{\left(4 - \nu + \sqrt{\nu^2 - 8\nu + 16 + 8\beta^2\nu - 32\beta^2}\right)(\nu - 2)} \tag{29a}$$

$$\mu_M = -\frac{\beta\delta_M^2}{\nu - 2}. \tag{29b}$$

To calculate the convolution of these factors, we simply multiply their characteristic functions and take inversions, recommendly by using $\mathcal{FFT}$. This method is faster and more accurate than using Monte Carlo methods.

3.1.4 Further Suggestions

As this paper shows, the number of possible copulas are infinite. We could of course look at many more subclasses of the $\mathcal{GH}$ distribution and even set up a copula with the $\mathcal{GH}$ distribution itself. However, increased model complexity in terms of additional free parameters etc., drastically complicates calibration issues. Therefore, we have restricted ourself to use the most popular hyperbolic distributions in finance and kept the number off free variables to three.

4 Loss Distribution Calculation

In this paper we will only calibrate our battery of copulas to homogenous credit portfolios, e.i. portfolios where all underlying credits have equal nominals and are assumed to have equal constant recovery rates. The problem of calculating the loss distribution of a homogenous credit portfolio is equivalent with calculating the number of defaults distribution of that portfolio. This follows since every default will result in the same amount of additional loss $\Delta_L = A(1 - \varrho)$, where A denotes the nominal of each credit and ϱ denotes the recovery rate. To compute the number of defaults distribution we use the probability generating function, as suggested by Laurent and Gregory (2005).

Assume a portfolio consisting of n credits. The loss distribution, denoted Υ, will then be a discrete distribution, taking values in $[0, L_{max}]$, where $L_{max} = n \times \Delta_L$. We will denote the default counting process of that portfolio by

$$N(t) = \sum_{i=1}^{n} N_i(t), \quad N_i(t) = \mathbf{1}_{\{\tau_i \leqslant t\}}, \tag{30}$$

where N_i are Bernoulli random variables. With this notation in hand we get that the probability generating function of $N(t)$ can be written as

$$\begin{aligned}\psi_{N(t)}(u) &= \mathrm{E}\left[\mathrm{E}\left[\prod_{i=1}^{n} u^{N_i(t)}|V\right]\right] \\ &= \mathrm{E}\left[\prod_{i=1}^{n}\left(q_{t,i}^{d|M} + p_{t,i}^{d|M} u\right)\right] \\ &= \int_{-\infty}^{\infty}\prod_{i=1}^{n}\left(q_{t,i}^{d|m} + p_{t,i}^{d|m} u\right) f_M(m)dm,\end{aligned} \tag{31}$$

where $q_{t,i}^{d|M} = 1 - p_{t,i}^{d|M}$. Furthermore, we can rewrite our probability generating function as

$$\psi_{N(t)}(u) = \mathrm{E}\left[\theta_0(M) + \theta_1(M)u + ... + \theta_n(M)u^n\right],$$

where $\theta_k(M)$ is given by a formal expansion of $\prod_{i=1}^{n}\left(q_{t,i}^{d|M} + p_{t,i}^{d|M} u\right)$. Thus, we get that the probability of k names being in default at time t is given by

$$\mathbb{Q}[N(t) = k] = \mathrm{E}[\theta_k(M)] = \int_{-\infty}^{\infty}\theta_k(m)f_M(m)dvm. \tag{32}$$

Accordingly, the probability of an equivalent loss rate is shown to be

$$\mathbb{Q}[\Upsilon(t) \leqslant l \times \Delta_L] = \sum_{k=0}^{l}\mathbb{Q}[N(t) = k], \quad l = 0, 1, ..., n. \tag{33}$$

To calculate these probabilities we need a method to get the value for each $\theta_k(m)$. If we assume equal credit spreads on all the underlying assets, i.e. $p_{t,i}^{d|M} = p_t^{d|M}$ for all ß, the problem can be solved analytically by the binomial distribution. However, if the credit spreads are not equal, we need a more general methodology to get the values of our $\theta_k(v)$. We solve this problem by circular convolution, preferably by using $\mathcal{FFT}$ as suggested by Robertson (1992) and Melchiori (2004). Finally, we'd like to point out that since the probable application of alternative copulas is for bespoke tranche pricing, it could be desirable to use a method that can handle heterogeneous portfolios. For an alternative $\mathcal{FFT}$ method that can incorporate heterogeneity, we refer the reader to Debuysscher and Szegö (2003).

4.1 Loss Distribution Calculation by LHP Approximation

In a large homogenous portfolio, with equal credit spreads for all the underlying assets, it is possible to do an approximation of the loss distribution by using the law of large numbers. This so called LHP approach, first suggested by Vasicek (1991), provides us with a closed form solution to the probability of different loss levels of a credit portfolio. A closed form solution to the loss distribution is of course very tractable. It simplifies calculations and increases computational speed since we get rid of some tiresome numerical integrations. Still, as suggested by Debuysscher and Szegö (2005), LHP approximations should only be used on portfolios with more than 500 credits and even with such large portfolios the method is not as accurate as other previously mentioned methods. As Vasicek deduced his result from the Gaussian one factor model, we will make a more general approach and calculate the LHP approximation of a Lévy one factor model.

As we mentioned in the previous section, the number of defaults distribution, given a homogenous portfolio with equal underlying credit spreads, is given by a binomial distribution. By making the substitution

$$s = F_\epsilon \left(\frac{F_X^{-1}\left(p_t^d\right) - \sqrt{\rho} m}{\sqrt{1-\rho}} \right),$$

we can write the number of defaults distribution as

$$\begin{aligned} \mathbb{Q}[N(t) = k] &= \int_{-\infty}^{\infty} \binom{n}{k} \left(p_t^{d|m}\right)^k \left(q_t^{d|m}\right)^{n-k} f_M(m) dm \\ &= -\int_0^1 \binom{n}{k} s^k (1-s)^{n-k} d\gamma(s), \end{aligned} \tag{34}$$

where

$$\gamma(s) = F_M \left(\frac{F_X^{-1}\left(p_t^d\right) - \sqrt{1-\rho} F_\epsilon^{-1}(s)}{\sqrt{\rho}} \right).[3] \tag{35}$$

Further, we have that

$$\mathbb{Q}[N(t) \leqslant l] = \sum_{k=0}^{d_r n} -\int_0^1 \binom{n}{k} s^k (1-s)^{n-k} d\gamma(s), \tag{36}$$

where $d_r = l/n$, $0 \leqslant d_r \leqslant 100\%$, denotes the portfolio default rate. Now, by the law of large numbers, we notice that

$$\lim_{n\to\infty} \sum_{k=0}^{d_r n} \binom{n}{k} s^k (1-s)^{n-k} = \begin{cases} 1, & d_r > s \\ 0, & d_r < s \end{cases}, \tag{37}$$

which gives us that equation (36) can be expressed as

$$\mathbb{Q}[N(t) \leqslant l] = -\int_0^1 \mathbf{1}_{\{s<d_r\}} d\gamma(s) = 1 - \gamma(d_r). \tag{38}$$

□

The LHP approximation is highly negatively skewed and has higher kurtosis compared to the more accurate $\mathcal{FFT}$ method, see Figure 1.

[3] Note that $F_M(-x) \neq 1 - F_M(x)$ if the distribution of M is skewed. Therefore we can't rewrite our $\gamma(s)$ as in Vasicek (1991).

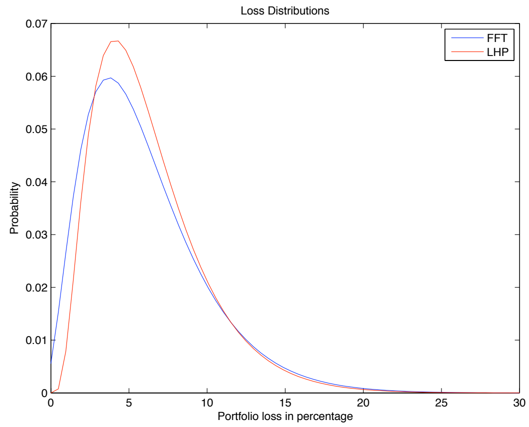

Figure 1: Loss distributions using a Gaussian copula with 125 names, $p_i^d = 0.1$ for all i, $\rho = 0.1$, $r = 3\%$ and $\varrho = 0.4$.

5 Arbitrage Pricing of CDOs

Assume a portfolio consisting of n credits. Given the loss distribution of that portfolio, we can create a default swap for some given loss interval $[A, B]$, where $0 \leqslant A < B \leqslant L_{max}$. Every such default swap is called a CDO tranche and every CDO consists of one or more tranches. The spread for each such tranche is set so that

$$PL + AL = DL, \tag{39}$$

where PL, AL and DL denote the premium leg, accrued leg and default leg respectively. In Table 1 we present the CDO structure traded on iTraxx

Table 1: The CDO structure of iTraxx

Tranche Limits	Tranche Name
0-3%	Equity
3-6%	Junior Mezzanine
6-9%	Senior Mezzanine
9-12%	Senior
12-22%	Super Senior

5.1 Pricing the Premium Leg

We let the payment dates follow an equidistant time grid $t_0, t_1, ..., t_K$, where t_0 is today, $t_K = T$ is the maturity date and $\Delta_t = t_i - t_{i-1}$. The discount factor is given by

$B(t_i)$, the forward interest rate is assumed to be deterministic and the premium rate is denoted r_p. Further, we let $\mho_{[A,B]}(t_i)$ denote the cumulative loss of the CDO tranche at time t_i. Thus, if the total nominal of the portfolio is equal to 1, we have that

$$\mho_{[A,B]}(t) = 0, \qquad \Upsilon(t) \leqslant A \tag{40a}$$

$$\mho_{[A,B]}(t) = \Upsilon(t) - A, \qquad A < \Upsilon(t) \leqslant B \tag{40b}$$

$$\mho_{[A,B]}(t) = B - A, \qquad \Upsilon(t) > B. \tag{40c}$$

We denote the outstanding nominal of a tranche by $\mho_{[A,B]}(\infty) - \mho_{[A,B]}(t)$, where $\mho_{[A,B]}(\infty)$ denotes the initial nominal. Accordingly, the premium leg can be written as

$$\Pi^{PL}(t_0) = \sum_{i=1}^{K} r_p B(t_i) \Delta_t \mathrm{E}\left[\mho_{[A,B]}(\infty) - \mho_{[A,B]}(t_i)\right]. \tag{41}$$

To calculate $\mathrm{E}\left[\mho_{[A,B]}(t_i)\right]$, we need to know the distribution of Υ at every time t_i. Since this is a discrete distribution, we need to be careful if the tranche limits lay between two possible loss levels. We let $a = A/\Delta_L$, $b = B/\Delta_L$ and if a or b is not an integer, we let $a_u, b_u = \inf\{n \in \mathbb{N} | n > a, b\}$. By letting $f_{\Upsilon(t_i)}$ denote the probability function of Υ for the different payment dates, we have that

$$\begin{aligned} \mathrm{E}\left[\mho_{[A,B]}(t_i)\right] =& (B - A) \sum_{m=b_u}^{N_{max}} f_{\Upsilon(t_i)}(m\Delta_L) \\ &+ \Delta_L (a_u - a) f_{\Upsilon(t_i)}(a_u \Delta_L) \\ &+ \sum_{m=a_u+1}^{b_u-1} (m\Delta_L - A) f_{\Upsilon(t_i)}(m\Delta_L). \end{aligned} \tag{42}$$

This function is applicable for all cases of a and b, since if a is an integer the interpolation term will equal zero. Now, consider the special case of a LHP approximation. As we showed in section 4.1, the loss distribution provided by a LHP approximation is continuous. Thus, the expected loss for each tranche can be written as

$$\mathrm{E}\left[\mho_{[A,B]}(t_i)\right] = (1 - \varrho) \int_{\frac{A}{(1-\varrho)}}^{\frac{B}{(1-\varrho)}} \gamma(s) ds. \tag{43}$$

5.2 Pricing the Accrued Leg

We let $t_{i(k)}$ denote the last payment date before the k-th default time τ_k. We realize that the k-th default will only result in an increase of $\mho_{[A,B]}$ if $A < \Upsilon(\tau_k) \leqslant B$. This possible increase is given by $\mho_{[A,B]}(\tau_k) - \mho_{[A,B]}(\tau_k^{-1})$, where $\mho_{[A,B]}(\tau_k^{-1})$ is the value of the cumulative loss at time τ_k had the k-th default not happened. Thus, the accrued leg can be written as

$$\mathrm{E}\left[\sum_{k=1}^{n} r_p \mathbf{1}_{\{\tau_k \leqslant T\}} B(\tau_k)(\tau_k - t_{i(k)}) \Big(\mho_{[A,B]}(\tau_k) - \mho_{[A,B]}(\tau_k^{-1})\Big)\right]. \tag{44}$$

Now, since the cumulative loss function of the CDO tranche is strictly increasing, we can define Stieltjes integrals with respect to $\mho_{[A,B]}(t)$. This gives us that equation (44) can also be written as

$$\mathrm{E}\left[\sum_{i=0}^{K-1} r_p \int_{t_i}^{t_{i+1}} B(s)(s-t_i)d\mho_{[A,B]}(s)\right], \tag{45}$$

which, by integration by parts, finally gives us that

$$\begin{aligned}\Pi^{AL}(t_0) = \sum_{i=0}^{K-1} r_p \Bigg(& \mathrm{E}\left[\mho_{[A,B]}(t_{i+1})\right] B(t_{i+1})\Delta_t \\ & - \int_{t_i}^{t_{i+1}} \mathrm{E}\left[\mho_{[A,B]}(s)\right] B(s)\Big(1 - f_r(t_0,s)(s-t_i)\Big)ds\Bigg).\end{aligned} \tag{46}$$

5.3 Pricing the Default Leg

There is a default payment for every default that effects the CDO tranche. Now, having priced the accrued leg, the pricing of the default leg is straightforward. We have that

$$\begin{aligned}\Pi^{DL}(t_0) &= \mathrm{E}\left[\sum_{k=1}^{n} \mathbf{1}_{\{\tau_k \leqslant T\}} B(\tau_k)\Big(\mho_{[A,B]}(\tau_k) - \mho_{[A,B]}(\tau_k^{-1})\Big)\right] \\ &= \mathrm{E}\left[\int_{t_0}^{T} B(s)d\mho_{[A,B]}(s)\right] \\ &= \mathrm{E}\left[\mho_{[A,B]}(T)\right] B(T) + \int_{t_0}^{T} \mathrm{E}\left[\mho_{[A,B]}(s)\right] f_r(t_0,s)B(s)ds.\end{aligned} \tag{47}$$

6 Copula Calibrations and Results

Our proposed copula models are calibrated for market data of June 2006, which held a total of 22 trading days this year. We have used tranche mid-spreads and the equity tranche is noted in upfront plus $500bp$ running spread.[4] Creditcurves are calculated by bootstrapping the fair 3 and 5 year index mid-spreads and the interest rate is set to 3%. Distribution parameters are optimized by minimizing the sum of the quadratic errors for each tranche.

In Table 2, we show the results of our copula calibrations using $\mathcal{FFT}$ to calculate the loss distributions. The mean absolute errors for each tranche noted and the average iTraxx prices are presented just to give the reader an idea of the market prices for this particular month. As we can see, all hyperbolic copulas are far superior to the Gaussian. In Table 3, we present the results using LHP to calculate the loss distributions. Evidently, the $\mathcal{FFT}$ method, which in theory should be more exact, yields better results for the hyperbolic models whereas the LHP method yields better results for the Gaussian copula. Alas, the increased negative skewness and kurtosis gained using LHP

[4] $PL(500bp) + AL(500bp) + Upfront \times (B - A) = DL$

help to improve the Gaussian copula. The calibrated parameters of the hyperbolic copulas are shown in Table 4 and the implied factor distributions are plotted in Figures 2 and 3. We note that the factor distributions calibrated using LHP have less skewness and kurtosis as to compensate for the above mentioned LHP properties.

	$\mathcal{FFT}$ Market Fit of June 2006					$\sum \overline{AE}$
TL	0-3%	3-6%	6-9%	9-12%	12-22%	
$\overline{\text{iTraxx}}$	23.6%	77.9	21	9.3	4.1	
$\overline{AE_{\Phi}}$	0.1%	44.9	6	7	4	71.9
$\overline{AE_{\mathcal{NIG}}}$	0%	0.3	1.8	2.3	1.2	5.6
$\overline{AE_{\mathcal{VG}}}$	0%	0.5	0.8	0.8	1.2	3.3
$\overline{AE_{\mathcal{S}t}}$	0%	0.2	2.8	1.1	0.7	4.8

Table 2: Mean absolute errors using $\mathcal{FFT}$.

	LHP Market Fit of June 2006					$\sum \overline{AE}$
TL	0-3%	3-6%	6-9%	9-12%	12-22%	
$\overline{\text{iTraxx}}$	23.6%	77.9	21	9.3	4.1	
$\overline{AE_{\Phi}}$	0%	39.5	3.7	5.6	3.8	52.6
$\overline{AE_{\mathcal{NIG}}}$	0%	0.4	4.5	3.3	1.2	9.4
$\overline{AE_{\mathcal{VG}}}$	0%	0.7	4.3	2.8	1	8.8
$\overline{AE_{\mathcal{S}t}}$	0%	0.1	2	1.9	1	5

Table 3: Mean absolute errors using LHP.

7 Conclusions

The proposed one factor Lévy model provides a consistent mathematical framework that offers a large variety of resourceful options for portfolio risk modelling. As previously shown in the literature, this paper confirms that negative skewness and high kurtosis are of major importance for depicting how the market distributes credit risk. In particular, this paper validates the battery of copulas by applying both LHP approximations as well as $\mathcal{FFT}$ methods to a larger set of data. The fact that the more adequate $\mathcal{FFT}$ approach yields better results for the hyperbolic models and worse for the Gaussian model is of significant interest. It argues just as strongly for the ability of hyperbolic copulas to capture market behavior, as it does against the likeliness of Gaussian distributed risk factors. However, even though we achieve persuading results, we understand the difficulty of motivating the increased model complexity to industry practitioners. Nevertheless, we strongly argue that the proposed copulas should be used as a benchmark tool to the base correlation approach of pricing bespoke CDO tranches.

	Distribution Parameters of June 2006					
$\mathcal{FFT}$	$\overline{\rho}$	$\overline{\alpha}, \overline{\nu}$	$\overline{\beta}$	V$[\rho]$	V$[\alpha]$, V$[\nu]$	V$[\beta]$
$\mathcal{NIG}$	0.0986	0.659	-0.1536	0.0002	0.0118	0.0095
$\mathcal{VG}$	0.0856	1.423	-0.723	0.0001	0.0143	0.0168
$\mathcal{S}t$	0.1124	4.439	-0.7349	0.0033	0.4925	0.3747
LHP	$\overline{\rho}$	$\overline{\alpha}, \overline{\nu}$	$\overline{\beta}$	V$[\rho]$	V$[\alpha]$, V$[\nu]$	V$[\beta]$
$\mathcal{NIG}$	0.1243	0.8794	-0.0797	0.0001	0.0127	0.0242
$\mathcal{VG}$	0.1123	1.174	-0.1462	0.0001	0.0219	0.0069
$\mathcal{S}t$	0.108	5.112	-0.2583	0.0016	4.006	0.1413

Table 4: Mean and variance of distribution parameters.

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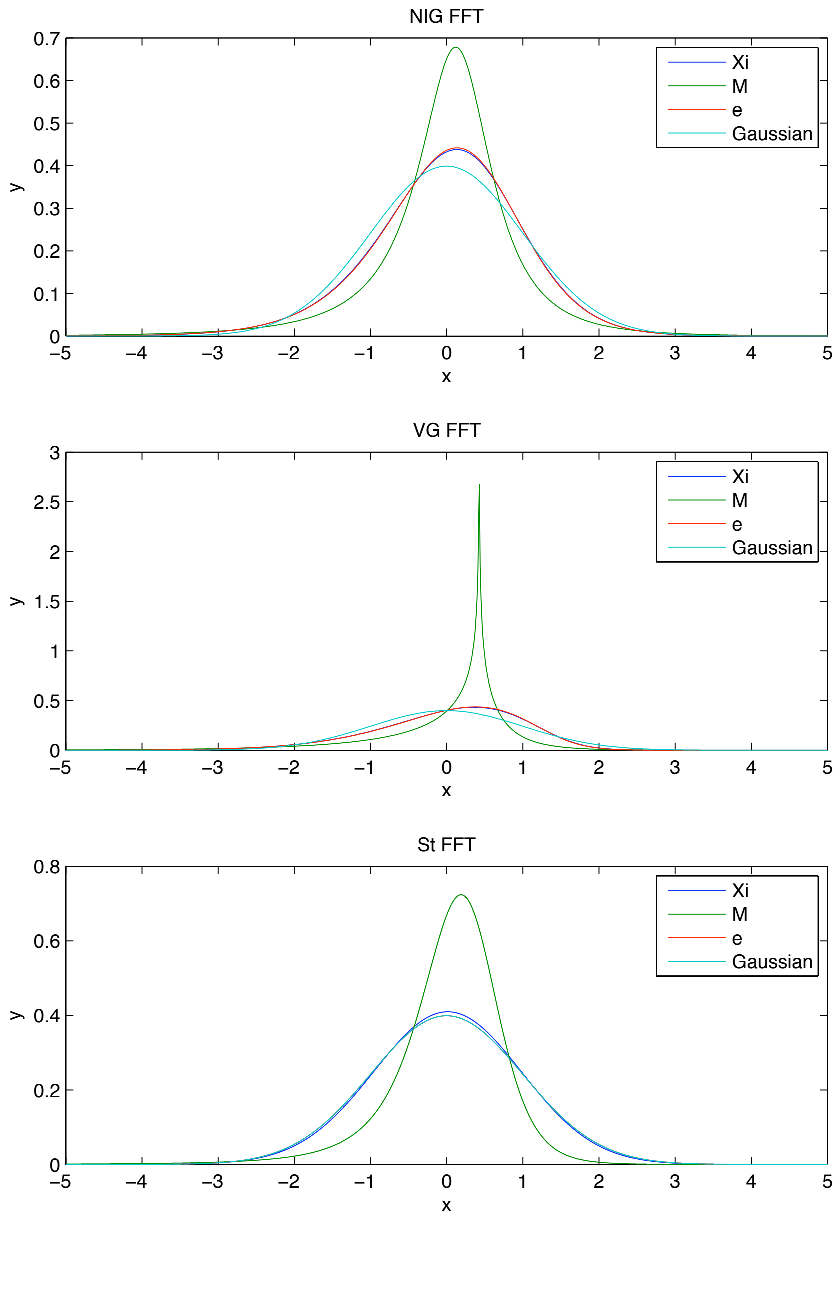

Figure 2: Distribution plot of the factor model, $X_i = \sqrt{\rho}M + \sqrt{1-\rho}\epsilon_i$, given the average parameters in Table 4

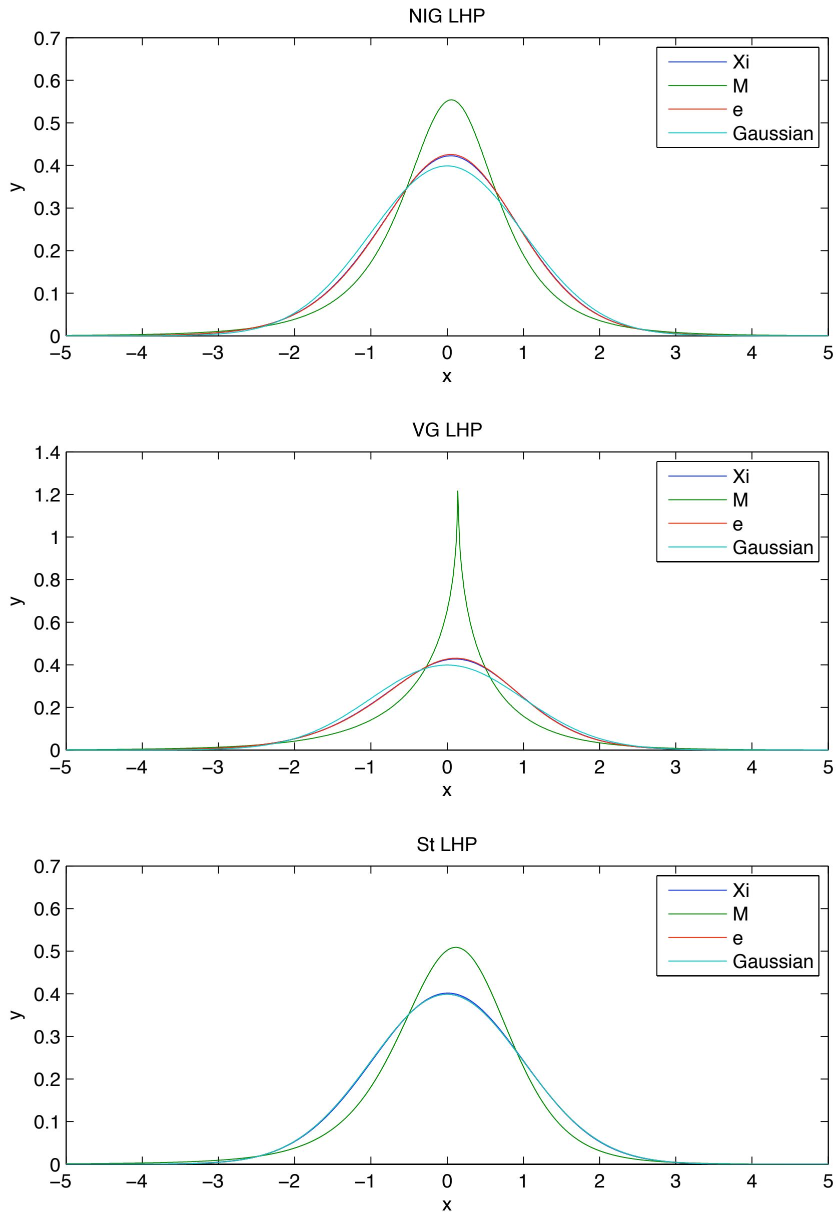

Figure 3: Distribution plot of the factor model, $X_i = \sqrt{\rho}M + \sqrt{1-\rho}\epsilon_i$, given the average parameters in Table 4